Quantum Information Science

Summary
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TECHNISCHE
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1 Introduction to Quantum Information Science

What is Quantum Information?

- Quantum Information is that kind of information, which is carried by quantum systems from the preparation device to the measuring apparatus in a quantum mechanical experiment.

What is information in a classical sense?

- The amount of knowledge we gain after learning the answer to a probabilistic question.
    - Consider a fair coin. You've no information about the outcome of the next flip and have to make a random guess. If someone were to tell you the outcome of the next flip, you would gain one bit of information(Since 0,1 represents Heads, Tails.
    - What if we have a biased coin where tails is more likely than heads? Learning Tails as an outcome surprises you less. You learn less, but how much less?
    - **Shannon Binary Entropy:** $h(p) = -p \cdot \log(p) - (1-p) \cdot \log(1-p)$

What is a measure of Quantum Information?

- Can we have an analogous measure in Quantum Information as in the classical one, when we no longer have just classical bits, light switches, coins, etc.?

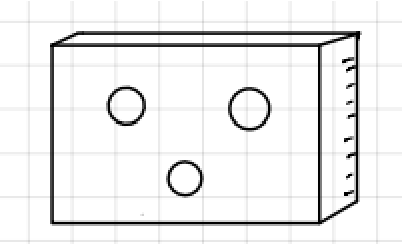
  The physical notion of the qubit is straightforward (once you know quantum theory), however, here we want to understand the information notion of a qubit, as in Shannon sense.

Classically, we qualified information by the amount of knowledge we gain after learning the answer to a probabilistic question. How can we reason about the knowledge of a quantum state?

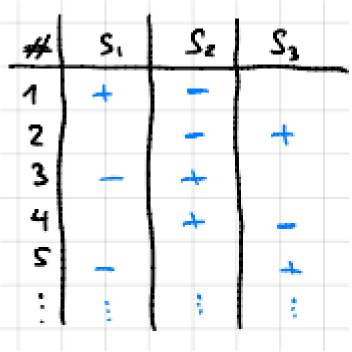
Answering this question in depth will be the primary goal of this course. In quantum physics we meet a few very important phenomena, which are either negligible or entirely absent in the classical limit(e.g. superposition). As a consequence, we shall not expect that classical notions in information theory can be just directly adopted to the quantum one.

1.1 Contextuality

- imagine a box with three holes (picture below)



- we can only look through two holes at the same time
- we look through every possible combination of the holes and notate the results
- every time we look through the hole, we can see a light flashing in exactly one hole while the other side remains dark(+ = light, - = dark).



#	S_1	S_2	S_3
1	+	−	
2		−	+
3	−	+	
4		+	−
5	−		+
⋮	⋮	⋮	⋮

What can we learn from these correlations?

- If such a scenario would ever occur, it is incompatible with our classical assumptions of physical properties existing independently of our observations.
- Assumption(classicality): Data can be thought of as originating from a process where
  - The first(or nature) generates the full table where every entry is defined.
  - Then we decide which pair to uncover.
- In every round k, we observe a *random* pair:
  - $\mathbb{P}$(signs are equal)$\geq \frac{1}{3} \Rightarrow$ Incompatible with the data set!
  - Physical quantities do not exist prior to us uncovering their values

1.2 Impossible machines

- Consider the universe in which experiments can give *contextual results*.
- From the box experiment, we can conclude that **measurements do not reveal pre-existing values**.

2 Linear algebra preliminaries

2.1 Complex linear algebra

2.2 Hilbert Spaces

2.3 Special types of matrices

2.4 Tensor product

3 Noiseless Quantum Theory

3.1 Quantum States

- Quantum mechanics: tells us that we can associate to a physical system (e.g. room) a corresponding Hilbert space, known as the **state space**.
- ***Postulate (Quantum State):*** Any closed system can be associated with a Hilbert Space $\mathcal{H}$. The state of the system is then completely described by its **state vector**, $|\psi\rangle = \sum\limits_{i=0}^{d-1} c_i|i\rangle$, with $\sum\limits_{i=0}^{d-1} c_i = 1$ and $|i\rangle_{i=0}^{d-1}$ forms a basis of $\mathcal{H}^d$.
- Instead of writing out the complex coefficients c_0 and c_1, we can also parameterize an arbitrary superposition with angles $\alpha, \beta, \theta \in \mathbb{R}$:

$$|\psi\rangle = e^{i\alpha}(\text{cos}\alpha\ |0\rangle + \text{e}^{\text{i}\beta}\cdot\text{sin}\alpha\ |1\rangle).$$

- However, the global phase $e^{i\alpha}$ vanishes in all important calculations as $e^{i\alpha}e^{-i\alpha} = 1$.

$$|\psi\rangle = \text{cos}\alpha\ |0\rangle + \text{e}^{\text{i}\beta}\cdot\text{sin}\alpha\ |1\rangle.$$

- Alternatively, we can say that the length of the state vector $|\psi\rangle$ is 1.
- This constraint is what makes this complex vector a quantum superposition, as it corresponds to the sum of probabilities of getting an outcome 'i'.
- This is also very straightforward to see:

$$\langle \psi \,|\, \psi\rangle = \big(cos(\alpha)\langle 0| + e^{-i\beta}\cdot sin(\alpha)\langle 1|\big)\,\big(cos(\alpha)|0\rangle + e^{i\beta}\cdot sin(\alpha)|1\rangle\big)$$

$$= cos^2(\alpha)\underbrace{\langle 0\,|\,0\rangle}_{=1} + e^{i\beta}\cdot sin(\alpha)cos(\alpha)\underbrace{\langle 0\,|\,1\rangle}_{=0} + e^{-i\beta}\cdot sin(\alpha)cos(\alpha)\underbrace{\langle 1\,|\,0\rangle}_{=0} + sin^2(\alpha)\underbrace{\langle 1\,|\,1\rangle}_{=1}$$

$$= cos^2(\alpha) + sin^2(\alpha) = 1.$$

3.2 Quantum measurements

We will now discuss the postulate of QM which is concerned with measurements. The central result is that measuring a quantum system in inherently probabilistic, i.e. the outcome of a measurement is not deterministic and truly random. For any $|\psi\rangle$, the probability of measuring an outcome v_i is given by the absolute-square of the inner product between the "measurement state"$|v_i\rangle$ and the state $|\psi\rangle$:

$$\mathrm{P}(i) = |\langle v_i \mid \psi\rangle|^2.$$

The value of the inner product is called the probability amplitude and can be negative or even complex. Immediately after the measurement, the state $|\psi\rangle$ collapses into a post-measurement state $|\psi'\rangle$. This post-measurement state is:

$$|\psi\rangle' = \frac{M_i|\psi\rangle}{N_i}$$

where $M_i = |v_i\rangle\langle v_i|$ and $N_i = \sqrt{P(i)}$ are the measurement operator and normalizing constant, respectively. These results are digested in the following postulate.

Postulate (Quantum State): Quantum measurements are described by a collection of measurement operators M_i, where i indicated the outcome of the experiment. Let $|\psi\rangle$ be the state before the measurement, then the state immediately after the measurement is $|\psi'\rangle$= $(M_i|\psi\rangle)/N_i$ where $N_i = \sqrt{P(i)}$ is for normalization.

More generally, a quantum measurement process is described by a Hermitian matrix, its eigenvalues and eigenvectors.Given a qudit state $|\psi\rangle$, we say that we measure an observable M (e.g. position, polarization, momentum) on it, and we associate measurement outcomes to eigenvalues of M, the post-measurement states to eigenvectors, and probabilities to inner-products.

$$\mathrm{M}\,|v_i\rangle = \lambda_i\,|v_i\rangle, \qquad \forall\lambda_i \in \mathbb{R} \text{ as } M^\dagger = M$$

Since $\{|v_i\rangle\}$ are eigenvectors, we have $\langle v_i \mid v_i\rangle = \delta_{i,j}$ (**Kronecker-Delta**)

Note that: $M = \sum_{i=0}^{d-1} \lambda_i |v_i X v_i|$

On the other hand, we have a completeness relation: $\sum_{i=0}^{d-1} |v_i X v_i|$= I

3.3 Evolution of noiseless Quantum States

The evolution of quantum states describes how they pass between states and is described by linear transformations **U**, also called gates. Given a quantum state $|\psi\rangle$, one can transform it to another quantum state $|\widetilde{\psi}\rangle$using a gate. We denote an application of U_1, and then U_2 to a state $|\psi\rangle$, as $U_2U_1|\psi\rangle$.

Postulate (State Evolution): The evolution $|\psi\rangle \rightarrow |\widetilde{\psi}\rangle$ of a closed physical system is described by a linear transformation $U^\dagger U = \mathbf{I_n}$.

3.3.1 Gates

The Identity Gate

This a very simple gate, which doesn't change anything at all. It preserves the state of the system as it is - it gives us the identity of the qubit.

$$\text{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Pauli Gates

These gates work by changing the direction of the vector $|\psi\rangle$ in either the x, y or z direction.

$$\text{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli X Gates is the classical NOT Gate, transforming $|0\rangle \rightarrow |1\rangle$. On the other hand the Paul Y Gate gets us the same results however instead moving through real space, we move through imaginary space instead. Lastly, the Pauli Z Gate changes the state of the qubit along the plane formed by the vector represented by our two states.This means that no change will occur if we're fully in one state or the other - only if we're somewhere between the two.

The Hadamard Gate

The Hadamard gate H transform the two basis states $|0\rangle$ and $|1\rangle$ into an equal superposition of themselves: $\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

$$\mathrm{H}|0\rangle = \mathrm{H}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle.$$

$$\mathrm{H}|1\rangle = \mathrm{H}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle.$$

As these states are extremely important, we often denote them as $|+\rangle$ and $|-\rangle$. In addition, these two states are again basis states of a Hilbert space as the following holds:

$$\langle + | + \rangle = 1, \quad \langle + | - \rangle = 0, \quad \langle - | - \rangle = 1, \quad \langle - | + \rangle = 0$$

Common qubit gates and their effect on the Pauli and Hadamard gate:

Gate U	U$\|0\rangle$	U$\|1\rangle$	U$\|+\rangle$	U$\|-\rangle$
$\mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{HZH}$	$\|1\rangle$	$\|0\rangle$	$\|+\rangle$	$\|-\rangle$
$\mathrm{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$i\|1\rangle$	$-i\|0\rangle$	$-i\|-\rangle$	$i\|+\rangle$
$\mathrm{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\|0\rangle$	$-\|1\rangle$	$\|-\rangle$	$\|+\rangle$
$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\|+\rangle$	$\|-\rangle$	$\|0\rangle$	$\|1\rangle$

3.4 Composite Systems

As in classical computing where we are concerned with more than one bit, QC also works with more than one qubit. The formalism for this are tensor products between the Hilbert spaces of the individual qubits. For example two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

This definition has the effect of applying unitary A to the first and unitary B to the second qubit in a tensor-multiplied Hilbert space, i.e.:

$$(A \otimes B)(|\psi_1\rangle \otimes |\psi_2\rangle) = (A|\psi_1\rangle) \otimes (B|\psi_2\rangle)$$

3.4.1 Quantum Entanglement

Definition: Given a quantum state $|\psi_{12}\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ of a composite system. We call it a product state if

$$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle, \text{ where } |\psi_1\rangle \in \mathbb{C}^{d_1} \text{ and } |\psi_2\rangle \in \mathbb{C}^{d_2}$$

Otherwise we call it entangled.

3.4.2 Resource Theory of entanglement

Given two quantum states $|\psi_1\rangle \in \mathbb{C}^{d_1}$ and $|\psi_2\rangle \in \mathbb{C}^{d_2}$ How do compare them/the entanglement contained in them?

Definition: We consider two states to be an identical resource if they can be reached from each other by local unitary transformations

$$\text{If } U_1 \otimes U_2|\psi_{12}\rangle = |\phi_{12}\rangle, \text{ we write } |\psi_{12}\rangle \overset{\text{LU}}{=} |\phi_{12}\rangle$$

Since unitaries are reversible, this defines equivalence classes of quantum states under LU transformations.

E.g. all the product states are in a single equivalence class, the Bell states are equivalent:

$$\mathbb{I} \otimes Z \cdot |\phi^+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = |\phi^-\rangle \Rightarrow |\phi^+\rangle \overset{\text{LU}}{=} |\phi^-\rangle$$

Theorem: (Schmidt decomposition) Suppose $|\psi_{12}\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is a “pure” state of a composite system. There exists ONB $|i_1\rangle$ for system 1 and a orthonormal basis $|i_2\rangle$ for system 2, such that:

$$|\psi_{12}\rangle = \sum_{i=0}^{k-1} \sqrt{\lambda_i} |i_1\rangle \otimes |i_2\rangle,$$

where $\sqrt{\lambda_i} > 0$ are called Schmidt coefficients satisfying $\sum \lambda_i = 1$. (k-1) is called the Schmidt rank of the state $|\psi_{12}\rangle$. Then it is easy to see, if we sort λ_i in descending order: Two bipartite states are LU equivalent iff their Schmidt coefficients coincide.

Example: Express the given state $|\psi\rangle \in \mathbb{C}^3 \times \mathbb{C}^3$ in its Schmidt decomposition

$$|\psi\rangle = \frac{1}{\sqrt{6}} \left(|00\rangle + \omega|01\rangle + \omega^2|02\rangle - |20\rangle - \omega|21\rangle - \omega^2|22\rangle \right)$$

where $\omega = e^{2\pi/3}$ is a root of unity.

Solution: To represent the state in its Schmidt decomposition we have to transform in such a way, that in the end it is a product state. As the first step, the $|0\rangle$ and the $|2\rangle$ can be pulled out.

$$\frac{1}{\sqrt{6}} \left(|0\rangle \otimes \left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle \right) - |2\rangle \otimes \left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle \right) \right)$$

Since $|0\rangle$ and $|2\rangle$ are being multiplied with the same term. We can represent the state as one tensor product.

$$\frac{1}{\sqrt{6}}\left((|0\rangle - |2\rangle) \otimes \left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)\right)$$

This now almost looks like a product state, however both states have to be normalized.

$$\frac{(|0\rangle - |2\rangle)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)}{\sqrt{3}}$$

We have now represented the state in its Schmidt decomposition. Since the first element of the ket vectors either takes the value of 0 or 2, we know that these vectors build an orthonormal basis for that system. The same analogy also applies to the second element of the ket vectors in $|\psi\rangle$. The Schmidt number for our case would then be 1.

3.4.3 Local operations and classical communication (LOCC)

Local unitary matrices classify 2-qubit quantum states in ∞ equivalence classes(If two states are equivalent under LU transformation, their Schmidt numbers are the same). However, one of the tasks in quantum information theory is to distill or concentrate the entanglement of a given state. Does there exist a broader physically motivated transformation which could help us rank states with reference to the entanglement in them? For this there are basically two different protocols:

1. non-local quantum measurements on many copies of the initial state or
2. local operations with possible classical communication(LOCC), performed on only one copy of the state.

Since local measurements can be performed more easily than non-local ones, the 2nd option is of special interest. From the point of view of quantum communication LOCC protocols are important because there is no perfect communication channel in the real-world. Hence it is natural to ask how much entanglement can be obtained from the imperfectly entangled states which arise, for example, during the sharing of a perfectly entangled state between two observers using only LOCC.

Example: LOCC operations

Physical processes which are involved in LOCC operations become more plausible by considering the following simple example. Consider two observers, Alice and Bob, who share the two Bell states:

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B)$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

and are provided some classical communications channel (a phone or internet). Alice and Bob can choose one of the shared states, but the information about which state it is exactly is lacking. By using LOCC, Alice and Bob can distinguish between these two states. To do so, Alice has just to measure her qubit and send the measurement outcome to Bob. After receiving this, Bob has to perform a measurement on his qubit, after which Alice and Bob would certainly know which state they had. If, for example, Alice would measure 0 and Bob would measure 1, then they measured $|\psi^-\rangle$.

Majorization

An bipartite state $|\psi\rangle$ can transform to a different quantum state $|\phi\rangle$ using LOCC iff their Schmidtvalues λ_ψ are majorized by λ_ϕ. But what does majorized mean?

Suppose X = $(x_1, \dots, x_d)$ and Y = $(y_1, \dots, y_d)$ are real d-dimensional vectors. Then X is majorized by Y, written X $\succ$ Y, if $\forall$ k in the range 1, . . . , d:

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$$

The $\downarrow$ indicates that the elements are to be taken in descending order. So, $x_1^{\downarrow}$ is the largest element in X.

In the two qubit case, the Bell state is majorized by any $|\psi_1\rangle$.

$$\lambda_{\phi^+} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \text{ and } |\psi_1\rangle = \left(cos^2(\theta), sin^2(\theta)\right)$$

For k = 1:

$$cos^2(\theta) > \frac{1}{2}$$

For k = 1:

$$cos^2(\theta) + sin^2(\Theta) = 2 \cdot \frac{1}{2}$$

This means that the Bell state tranforms to every state using LOCC.

Example:

Which of the following states can be converted to the other two using LOCC operations? Explain your result:

$$|\psi_1\rangle = \frac{1}{\sqrt{6}}(|10\rangle + \sqrt{2}|02\rangle + |21\rangle + |20\rangle - |11\rangle)$$
$$|\psi_2\rangle = \sqrt{\frac{2}{5}}|10\rangle + \frac{1}{\sqrt{5}}|02\rangle + \sqrt{\frac{2}{5}}|21\rangle \; |\psi_3\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle$$

Step 1: Transform the states into their Schmidt decomposition.

3.4.4 Three quantum protocols

In this chapter, we study the fundamental, unit quantum communication protocols. These protocols involve a single sender Alice and a single receiver Bob. The protocols are ideal and noiseless because we assume that Alice and Bob can exploit perfect classical communication, perfect quantum communication, and perfect entanglement.

Any information-processing protocol that implements the above map simulates a noiseless qubit channel. We label the communication resource of a noiseless qubit channel as follows:

$$[\text{q} \rightarrow \text{q}]$$

where the notation indicates one forward use of a noiseless qubit channel. A noiseless classical bit channel is any mechanism that implements the following map:

$$|i\rangle\langle i|_A \Rightarrow |i\rangle\langle i|_B$$
$$|i\rangle\langle j|_A \Rightarrow 0 \text{ for i} \neq \text{j}$$

Classical communication

One bit of communication is transferred from Alice to Bob, denoted as:

$$[\mathrm{c} \to \mathrm{c}]$$

Quantum communication (Elementary Coding)

One bit of communication is transferred from Alice to Bob, denoted as:

$$[\mathrm{q} \to \mathrm{q}]$$

It is also possible to send classical information using a noiseless qubit channel. A simple protocol for doing so is *elementary encoding*. This protocol consists of the following steps:

1. Alice prepares either $|0\rangle$ or $|1\rangle$, depending on the classical bit that she would like to send.
2. She transmits this state over the noiseless qubit channel, and Bob receives the qubit.
3. Bob performs a measurement in the computational basis to determine the classical bit that Alice transmitted.

Elementary coding succeeds without error because Bob's measurement can always distinguished the classical states $|0\rangle$ and $|1\rangle$. The following resource inequality applies to elementary coding:

$$[\mathrm{q} \to \mathrm{q}] \geq [\mathrm{c} \to \mathrm{c}]$$

Shared entanglement (Entanglement Distribution)

The entanglement distribution protocol is the most basic of the three unit protocols. It exploits one use of a noiseless qubit channel to establish one shared noiseless ebit. It consists of the following two steps:

1. Alice prepares a Bell state locally in her laboratory. She prepares two qubits in the state $|0\rangle_A|0\rangle'_A$, where we label the first qubit as A and the second qubit A'. She performs a Hadamard gate on the qubit A to produce the following state:

$$\left(\frac{|0\rangle_A+|1\rangle_A}{\sqrt{2}}|0\rangle_{A'}\right).$$

She then performs a CNOT gate with qubit A as the source qubit and qubit A' as the target qubit. The state becomes the following Bell state:

$$|\phi^+\rangle_{AA'} = \left(\frac{|00\rangle_{AA'}+|11\rangle_{AA'}}{\sqrt{2}}\right)$$

1. She sends the qubit A' to Bob with one use of a noiseless qubit channel. Alice and Bob then share the ebit $|\phi^+\rangle_{AB}$.

The following resource inequality quantifies the non-local resources consumed or generated in the above protocol:

$$[\mathrm{q} \rightarrow \mathrm{q}] \geq [\mathrm{qq}]$$

where [q $\Rightarrow q$] denotes one forward use of a noiseless qubit channel and [qq] denotes a shared, noiseless ebit. The meaning of the resource inequality is that there exists as protocol that consumes the resource on the left in order to generate the resource on the right. The best analogy is to think of a resource inequality as a "chemical reactionlike formula, where the protocol is like a chemical reaction that transforms one resource into another.

Superdense coding

We now outline a protocol named super-dense coding. It is named as such because it has the striking property that noiseless entanglement can double the classical communication ability of a noiseless qubit channel. It consists of three steps:

1. Suppose that Alice and Bob share an ebit $|\phi^+\rangle_{AB}$. Alice applies one of the four unitary operations I, X, Z, XZ to her share of the above state. The state becomes one of the following four Bell states (up to a global phase), depending on the message that Alice chooses:

$$|\phi^+\rangle_{AB'}, \quad |\phi^-\rangle_{AB'}, \quad |\psi^+\rangle_{AB'}, \quad |\psi^-\rangle_{AB'}$$

2. She transmits her qubit to Bob with one use of the noiseless qubit channel.

3. Bob performs a Bell measurement (a measurement in the basis $\{|\phi^+\rangle_{AB'}, |\phi^-\rangle_{AB'}|\psi^+\rangle_{AB'}, |\psi^-\rangle_{AB'}\}$) to distinguish the four states perfectly - he can distinguish the states because they are all orthogonal to each other.

Thus, Alice can transmit two classical bits (corresponding to the four messages) if she shares a noiseless ebit with Bob and uses a noiseless qubit channel. The super-dense coding protocol realizes the following resource inequality:

$$[\mathrm{qq}] + [\mathrm{q} \to \mathrm{q}] \geq 2[\mathrm{c} \to \mathrm{c}]$$

Example: Consider, Alice and Bob share the maximally entangled state, $|\psi^+\rangle_{A_3B} \in \mathbb{C}^2 \times \mathbb{C}^2$ and Alice, in addition, in her lab has a state $|\phi\rangle_{A_1A_2}$ =cos$\theta|00\rangle_{A_1A_2}$+sin$\theta|11\rangle_{A_1A_2} \in \mathbb{C}^4, \theta \in [0, \frac{\pi}{4}]$.

To sum up, before they start the protocol, the state vector of the entire system looks as follows:

$$|\psi\rangle = |\phi\rangle_{A_1A_2} \otimes |\psi^+\rangle_{A_3B} \in \mathbb{C}^8_A \otimes \mathbb{C}^2_B.$$

Hence in total, Alice possesses three qubits.

a) Alice acts on the state $|\psi\rangle$ with the operator $(\mathbf{I}_{A_1} \otimes H_{A_2} \otimes \mathbf{I_{A_3}})\,(\mathbf{I_{A_1}} \otimes \mathrm{CNOT}_{A_2A_3})$. Calculate the action on this operator on the state $|\psi\rangle$

Solution:

$$(\mathbf{I_{A_1}} \otimes \mathrm{CNOT}_{A_2A_3})\,(|\phi\rangle_{A_1A_2} \otimes |\psi^+\rangle_{A_3B})$$

Firstly, let's write out the tensor product on the right side.

$$(\mathbf{I_{A_1}} \otimes \mathrm{CNOT}_{A_2A_3})\,\frac{1}{\sqrt{2}}\,(\mathrm{cos}\theta|00\rangle_{A_1A_2}(|01\rangle + |10\rangle)_{A_3B} + \mathrm{sin}\theta|11\rangle_{A_1A_2}(|01\rangle + |10\rangle)_{A_3B})$$

We now apply the identity matrix **I** on qubit A$_1$, however since it's the identity it doesn't change anything. This means we can go straight to applying the CNOT gate on the qubits A$_2$ and A$_3$. The CNOT gate always operates on two qubits, flipping the value of second qubit (the target qubit) if the value of the first qubit(control qubit) is equal to 1. In our context this means that we have to check the value of the qubit A$_2$ and if its 1, we flip the value of qubit A$_3$. This transforms our state to:

$$\frac{1}{\sqrt{2}}\,(\mathrm{cos}\theta|00\rangle_{A_1A_2}(|01\rangle + |10\rangle)_{A_3B} + \mathrm{sin}\theta|11\rangle_{A_1A_2}(|\mathbf{1}1\rangle + |\mathbf{0}0\rangle)_{A_3B})$$

Before applying the second part of the operator, lets simplify our current result. To do this we factor in the $\frac{1}{\sqrt{2}}$ term. Which leaves us with:

$$\cos\theta|00\rangle_{A_1A_2} \underbrace{\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{A_3B}}_{=|\psi^+\rangle} + \sin\theta|11\rangle_{A_1A_2} \underbrace{\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)_{A_3B}}_{=|\phi^+\rangle}$$

Now lets apply the second part of the operator:

$$(\mathbf{I}_{A_1} \otimes \mathbf{H}_{A_2} \otimes \mathbf{I}_{A_3})\,(\cos\theta|00\rangle_{A_1A_2}|\psi^+\rangle + \sin\theta|11\rangle_{A_1A_2}|\phi^+\rangle)$$

Applying the identity again means the qubits A_1 and A_3 remain untouched. So we only need to apply the Hadamard gate on the qubit A_2. The Hadamard gate transform the ket vector $|0\rangle$ to $|+\rangle$ and the ket vector $|1\rangle$ to $|-\rangle$. This then leads us to the following result:

$$\cos\theta|0+\rangle_{A_1A_2}|\psi^+\rangle + \sin\theta|1-\rangle_{A_1A_2}|\phi^+\rangle$$

b) Suppose now Alice makes a measurement on the second and third qubit of the state obtained in (a), in the computational basis. Write down what the corresponding state would be for the subsystem A_1 and B in each case.

Solution: Firstly we write out the state from above. With $|+\rangle$ being equal $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle$ being equal $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. This leaves us with:

$$\cos\theta\frac{1}{\sqrt{2}}\left(|0001\rangle + |0101\rangle + |0010\rangle + |0110\rangle\right) + \sin\theta\frac{1}{\sqrt{2}}\left(|1000\rangle - |1100\rangle - |1011\rangle - |1111\rangle\right)$$

The first element of the ket vectors comes from the ket vector $|0+\rangle$, the second element represents the different values that the ket vector $|+\rangle$ or $|-\rangle$ takes and lastly the last two element come from the respective bell state. If we now look at the outcome for the qubits A_2 and A_3 we can see that we have 4 different combinations.

Outcome	Probability	A_2A_3	A_1B
00	$\frac{1}{4}$	$\|00\rangle$	cos$\theta\|01\rangle+$sin$\theta\|10\rangle$
01	$\frac{1}{4}$	$\|01\rangle$	cos$\theta\|00\rangle+$sin$\theta\|11\rangle$
10	$\frac{1}{4}$	$\|10\rangle$	cos$\theta\|01\rangle-$sin$\theta\|10\rangle$
11	$\frac{1}{4}$	$\|11\rangle$	cos$\theta\|00\rangle-$sin$\theta\|11\rangle$

c)Alice now communicates the result of her measurement outcomes to Bob so that he can 'correct' the final state accordingly. Write down the corrections which Bob has to make once he receives information about Alice's outcome.

Solution: We now need to compare our state $|\phi\rangle$ to the state in the A_1B column and look if we need to make any transformations to Bob's qubit **B** to change the state to the state $|\phi\rangle$. For example, in the first row Bob's qubit needs to be flipped to create the state $|\phi\rangle$, this why we need to apply a Pauli-X gate. In the second row no changes are neccessary. However, in the third row we encounter a similiar case to the one from the first row. But this time we also need to flip - to +. This is the job of the Pauli-Z gate. Lastly, in the last row we only need the Pauli-Z gate.

Outcome	Probability	A_2A_3	A_1B	Corrections
00	$\frac{1}{4}$	$\|00\rangle$	cos$\theta\|01\rangle+$sin$\theta\|10\rangle$	X
01	$\frac{1}{4}$	$\|01\rangle$	cos$\theta\|00\rangle+$sin$\theta\|11\rangle$	I
10	$\frac{1}{4}$	$\|10\rangle$	cos$\theta\|01\rangle-$sin$\theta\|10\rangle$	ZX
11	$\frac{1}{4}$	$\|11\rangle$	cos$\theta\|00\rangle-$sin$\theta\|11\rangle$	Z

3.4.5 Multipartite entanglement

The next step is to look at entanglement in N-qubits.
Defintion: A multipartite state $|\psi\rangle \in \mathcal{H}^{\otimes N}$ is full separable if it can be written as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$.

It is called biseparable if some of its grouped parties are entangled, e.g.

$$|\psi\rangle_{12|3\ldots n} = |\psi\rangle_{12} \otimes |\psi\rangle_{3\ldots N}.$$

If the state is not biseparable across any bipartition we say that it is genuine multipartite entangled.

3.4.6 Stochastic LOCC

When discussing LOCC, $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$, we requested that the transformation happens with probability 1. Let us now drop this requirement and ask that a state $|\psi\rangle$ can be transformed to the state $|\phi\rangle$ with nonzero success probability. We can use this tool to learn more about allowed LOCC. Moreover SLOCC are characterized much easier than LOCC.

Theorem: $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$, where $|\psi\rangle, |\phi\rangle \in \mathcal{H}_d^{\otimes N}$ if there exists $A_1, A_2, \ldots, A_n \in GL(d, \mathbb{C})$, such that$A_1 \otimes A_2 \otimes \cdots \otimes A_N |\psi\rangle = |\phi\rangle$. If we request all $A_i's$ to be invertible, than we are defining new equivalence classes.

E.g. In d-dimensional bipartite case, $|\phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle \xrightarrow{\text{LOCC}} |\psi_{12}^d\rangle \Rightarrow |\phi_d^+\rangle \xrightarrow{\text{SLOCC}} |\psi_{12}^d\rangle$ And as long $|\psi_{12}^d\rangle$ has the same Schmidt rank, we get equivalence.

$$|\phi_d^+\rangle \overset{\text{SLOCC}}{=} |\psi_{12}^d\rangle$$

3.4.7 Monogamy of entanglement

So how does entanglement scale in multiparticle systems? E.g. If we have three qubit systems can all three be maximally entangled?

In what follows, we argue that entanglement cannot be freely shared between arbitrarily many parties $\Rightarrow$ "Monogamy" of Quantum Entanglement.

Consider a three qubit state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 3}$. Show that if Alice and Bob exhibit perfect correlations in the computational and Hadamar bases, then Charlie's state must be in the tensor product with A's and B's state.

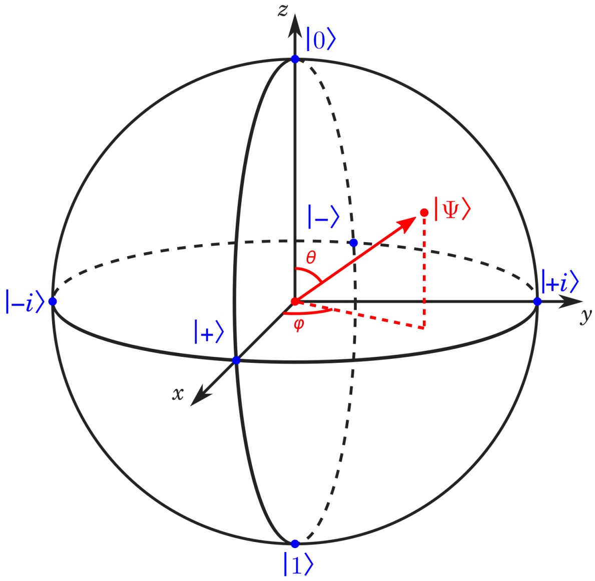
As a starting point, let us express a general three qubit state:

$$|\psi\rangle = \sum_{i,j,k} c_{i,j,k} |i, j, k\rangle, \sum_{i,j,k} |c_{i,j,k}|^2 = 1.$$

Since A and B shall exhibit perfect correlations in the computational basis, the outcome $|010\rangle, |011\rangle, |100\rangle$ and $|100\rangle$ shall never occur.

3.4.8 The Bloch Sphere

The Bloch sphere is a geometrical representation of the pure state space of two-level quantum mechanical system (qubit).



It's a handy device for visualizing these quantum states. Any sphere is a three dimensional object and therefore has an X, Y and Z axis. Even more important is that the surface of the Bloch sphere is the set of all pure quantum states, and the interior is the set of all mixed quantum states. As typical in geometry, we take the sphere to have a radius of 1 (to simplify the math). There is a unit vector or Bloch vector - the quantum state if the system - that moves around the inside of the sphere, either as the wave-function evolves over time or due to a measurement.

To better understand the Bloch sphere, we have to know a few things about quantum spin. Firstly, a spin or qubit measurement has two outcomes (eigenstates), which we call $|0\rangle$ and $|1\rangle$ These form the north and south poles (so to speak) of the sphere. This polar axis is canonically defined to be the Z-axis.A vital point here is that antipodal vectors, such as $|0\rangle$ and $|1\rangle$ are orthogonal states. Normally in geometry, orthogonal means two vectors (or even just lines) with a 90° angle, but in the Bloch sphere we define orthogonal to mean two vectors with a 180° angle.

Since the vectors $|0\rangle$ and $|1\rangle$ are defined as:

$$|Z_+\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |Z_-\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One can easily see that these vectors are orthogonal. Secondly, we use these two eigenstates as an eigenbasis that spans all possible states of our system. All other states are superpositions of these basis states.

In particular, we define the X-axis as:

$$|X_+\rangle = |+\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } |X_-\rangle = |-\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

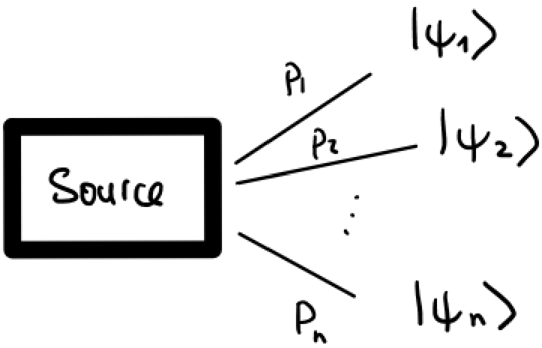
and the Y-axis as:

$$|Y_+\rangle = |+i\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } |Y_-\rangle = |-i\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

4 Noisy Quantum Theory

4.1 Noisy Quantum States

So far we only talked about pure quantum states. In a pure quantum state, the system is in a well-defined state described by a single wavefunction or ket vector. However this doesn't transfer very well to real world applications. Since we often don't know exactly which state our system is in. It could be in one of several possible states with certain probabilities. In order to describe a system like this we use mixed quantum states.A mixed quantum state is a way of describing this uncertainty or lack of knowledge. Instead of having a single ket vector, we have a mixture of ket vectors, each with its own probability of occurring. Note that in the former case, the vector moves smoothly and, per the wave-function, completely predictably. It is the latter case that gives quantum physicists the fits — measurement causes the vector to jump to an eigenstate, the dreaded "collapse" of the wave-function.



An ensemble of quantum states can then be represents as:

$$\mathscr{E} = \{p_X(x), |\psi_x\rangle\}_{x\in X}$$

Performing a measurement on the ensemble $\mathscr{E}$

Let Π_j be the element of this measurement and $\sum\limits_{j \in J} \Pi_j = 1$ and suppose the state was $|\psi_x\rangle$ when we a took the measurement. Then the conditional probability of obtaining an outcome j, when measuring $|\phi_x\rangle$ is $p_{J|X}(j|x) = \langle\psi_x|\Pi_j|\psi_x\rangle$.

5 Entropy & Information